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Theoretical approximation to transient heat conduction in nuclear waste repositories

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Abstract. Comparison is made between the numerical solution for temperature histories at a nuclear waste site and the analytical solution of a correlated thermal problem. The differences between both solutions are found. Both problems are nonlinear diffusion problems connected with the disposal of nuclear wastes.

1. Introduction

The safety repository of nuclear waste in a geologic formation implies considerations about the thermophysical properties and dimensions of the host rock as well as the depth of the burial, tectonic stability and the hydrologic properties of the rock [9]. From the thermal point of view, the first two considerations are the important ones. Moreover, the temperature in the near-field as in the far-field of a repository is probably the most important single parameter concerning the safe disposal of high-level waste. Therefore, the determination of temperature change is essential in the design and environmental impact of the facility. This determination implies a need to find a solution to the heat conduction equation.

The general thermal problem of a nuclear waste repository (NWR) is a time-dependent, nonlinear, non-homogeneous, multidimensional heat conduction problem. To predict the thermal field and the heat transfer rates from a NWR, several computer codes have been applied using finite-difference and finite-element methods [3], [7]. However, just a few analytical approaches to a NWR thermal problem have been made. Sweet and McCreight [8], based on reference [1], solve exactly the simplified one-dimensional semi-infinite linear Waste Isolation Pilot Plant (WIPP) problem. The lack of analytical approaches to the NWR thermal problem is due to the nonlinearity of the heat conduction NWR equation. The purpose of this study is to solve numerically the nonlinear one-dimensional NWR thermal problem and then, in an effort to solve analytically the thermal problem, a new nonlinear problem is defined and its analytical solution is compared with the numerical solution of the original problem.

2. Controlling equations

The one-dimensional, nonlinear model for the temperature distribution in a nuclear site is given by the following partial differential equation (for a definition of symbols, see Section 7:

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Nomenclature)

$$\frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) = C(T)\frac{\partial T}{\partial t} \tag{1}$$

where

$$T = T(x, t), \quad k(T) = k_0 F(T) \tag{2}$$

and

$$C(T) = \varrho(T)c_p(T) = C_0. \tag{3}$$

Boundary and initial conditions, $T(x, t)$:

1. $k(T)\dfrac{\partial R}{\partial x}(0, t) = -\ddot{q}(t)$, (subterranean site)
2. $T(l, t) = T_l$, (surface)
3. $T(x, 0) = T_0$.

The geometry and dimension of a NWR suggest that this model is reasonable, see [4] and [10].

The specific behavior of the thermophysical properties ($k = k(T)$ and $C(T) = C_0$ a constant, see [8] and [4]) in a NWR makes the one-dimensional nonlinear heat condution problem very difficult to solve analytically. However, if it is assumed that

$$C(T) = \varrho(T)c_p(T) = C_0 F(T), \tag{4}$$

then the problem defined by equations (1), (2), (4) and the boundary and initial conditions can be solved analytically, see [2]. Let us call this problem and the original one as problems A and B or cases A and B respectively.

3. Analytical solution

The transient problem is subject to a nuclear heat source of the form

$$\ddot{q}(t) = \ddot{q}_0 \, e^{-\lambda t}, \tag{5}$$

see [4], and the thermal conductivity is from [8],

$$k(T) = k_0 F(T) = k_0 (T_a/T)^p, \tag{6}$$

so using equations (4) and (6) gives

$$C(T) = C_0 (T_a/T)^p = \varrho_0 c_{p_0} (T_a/T)^p. \tag{7}$$

The transformation

$$V(T) = \int_{T_r}^{T} k(T)\,\mathrm{d}T, \tag{8}$$

which is the Kirchhoff transformation, is used to solve analytically the nonlinear problem. The exact solution is given by the following equation

$$T(x, t) = T_r\left[1 - \frac{p-1}{k_0 T_r} V(x, t)\left(\frac{T_r}{T_a}\right)^p\right]^{1/(1-p)}, \tag{9}$$

where $V(x, t)$ is the transformed temperature given by

$$V(x, t) = V_l + \ddot{q}_0 \frac{\sqrt{\alpha_0}}{\sqrt{\lambda}} \frac{\sin\sqrt{\lambda/\alpha_0}\,(l-x)}{\cos\sqrt{\lambda/\alpha_0}\,l} \mathrm{e}^{-\lambda t}$$

$$+ \frac{2}{l}\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\alpha_0 \gamma_n^2 t}}{(-1)^{n+1}}\left(\frac{V_0 - V_l}{\gamma_n}\cos\gamma_n x + \ddot{q}_0 \frac{\alpha_0 \sin\gamma_n(l-x)}{(\lambda - \alpha_0\gamma_n^2)}\right) \tag{10}$$

where

$$\gamma_n = \frac{(2n-1)\pi}{2l}, \quad n = 1, 2, 3, \ldots$$

and

$$V_l = \frac{k_0 T_r}{p-1}\left[1 - \left(\frac{T_r}{T_l}\right)^{p-1}\right]\left(\frac{T_a}{T_r}\right)^p \tag{11}$$

and

$$V_0 = \frac{k_0 T_r}{p-1}\left[1 - \left(\frac{T_r}{T_0}\right)^{p-1}\right]\left(\frac{T_a}{T_r}\right)^p. \tag{12}$$

Equation (9) with equation (10) represent the transient solution to the thermal problem A. To analyze the steady-state solution let us assume for simplicity that $T_r = T_a = T_l$, then $V_l = V_0 = 0$. The steady-state situation occurs when $t \to \infty$, so $\ddot{q}(t) \to 0$ and from equation (10), $V(x, \infty) = 0$. Substitution into (9) gives $T(x, \infty) = T_r = T_l = T_0$, which is the expected value.

4. Numerical solution

The problem defined by equations (1), (2), (3) and the initial and boundary conditions (Problem B) is solved numerically using a standard implicit scheme of the finite-difference

Table 1. NWR parameters, nuclear waste in salt.

Case A	
$C_0 = 2298\,\mathrm{kJ/m^3\,°K}$	$\alpha_0 = 74.92\,\mathrm{m^2/yr}$
Case B	
$C_0 = 1789.2\,\mathrm{kJ/m^3\,°K}$	$\alpha_0 = 96.24\,\mathrm{m^2/yr}$
Common parameters	
$k_0 = 5.46\,\mathrm{watts/m\,°K}$	$l = 660\,\mathrm{m}$
$p = 1.21$	$T_0 = T_l = T_a = T_r = 300°\mathrm{K}$
HLW nuclear waste	
$\ddot{q}_0 = 12\,\mathrm{watts/m^2}$	$1/\lambda^{\dagger} = 47.4\,\mathrm{yrs}$
TRU nuclear waste	
$\ddot{q}_0 = 0.7\,\mathrm{watts/m^2}$	$1/\lambda = 31250\,\mathrm{yrs}$

† with $\lambda = \ln 2/t_{1/2}$

method. Special care was taken with the boundary condition at $x = 0$. It was treated with a central difference of second-order approximation [5].

Table 1 gives a listing of the parameters used to plot the solutions. Two thermal load models were used. One corresponds to high-level waste (HLW) with $12\,\mathrm{w/m^2}$ as the initial thermal load and 32.9 years as the half-life. Another corresponds to transuranic nuclear waste (TRU) with $0.7\,\mathrm{w/m^2}$ and 21661 years as the initial thermal load and half-life, respectively.

5. Results

Using equations (9) and (10) and the values of Table 1, Fig. 1 was made, it shows the typical thermal behavior of a NWR. At $x = 0$, $\Delta T(= T(x, t) - T_l)$ starts from zero, then increases

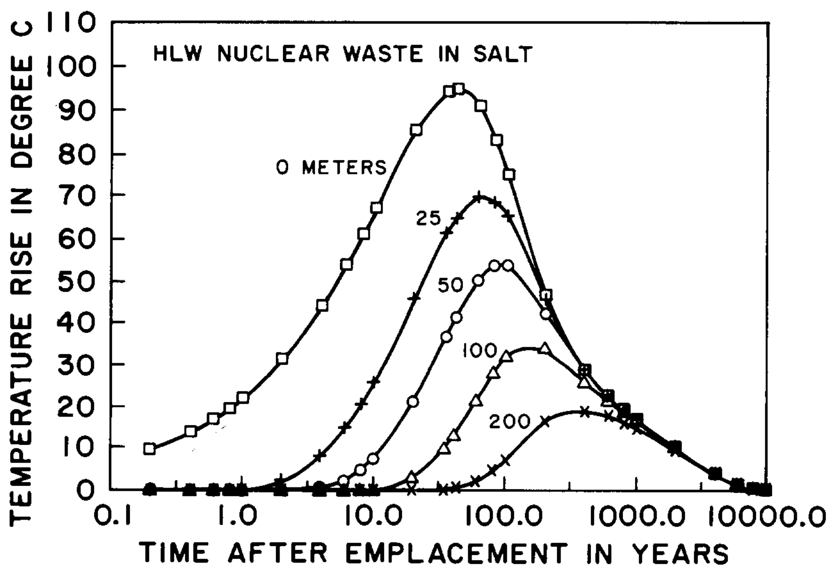

Fig. 1. Transient thermal response at several points above a NWR with an areal thermal load of $12\,\mathrm{w/m^2}$ (HLW) located 660 m deep in rock salt. Labels on curves indicate depth above NWR.

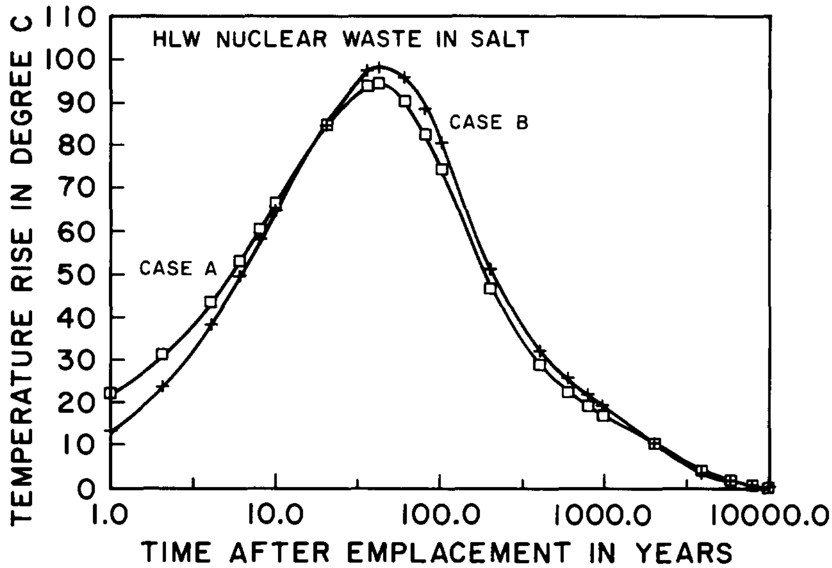

Fig. 2. Comparison of the exact solution of case A and the numerical solution of case B using HLW nuclear waste.

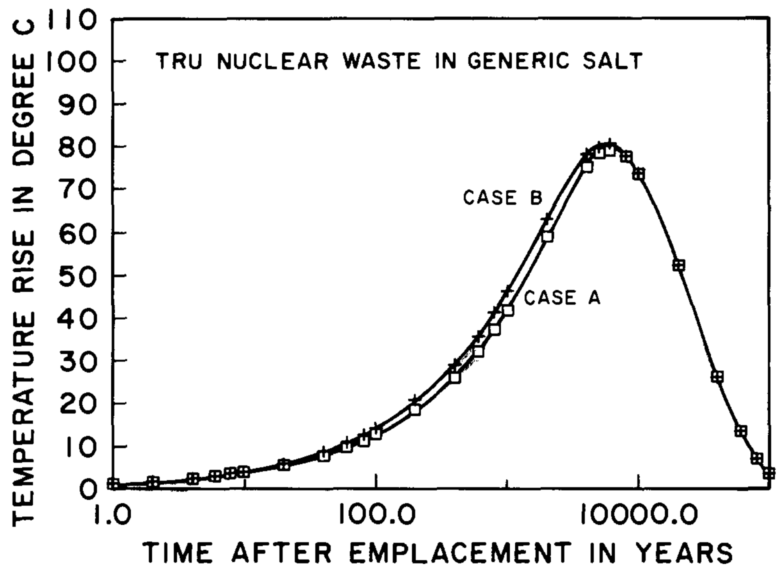

Fig. 3. Comparison of the exact solution of case A and the numerical solution of case B using TRU nuclear waste.

its value until it reaches a maximum and then decreases until it reaches zero again, this last stage corresponding to the steady-state situation. Also the figure shows that for the same time, ΔT decreases as x increases.

Figure 2 shows the solution of the problems A and B when HLW is used. It clearly shows the difference in the thermal response of the two problems. For case B, ΔT(max) was 98.7°C and for case A it was 94.6°C. In both cases the maximum occurs at 41 years. The difference represents an error of 4%. In Fig. 4 the differences between solutions were plotted. It shows that for times between 16 years and 2400 years, case A had a lower temperature than case B. Out of this time interval the inverse is true.

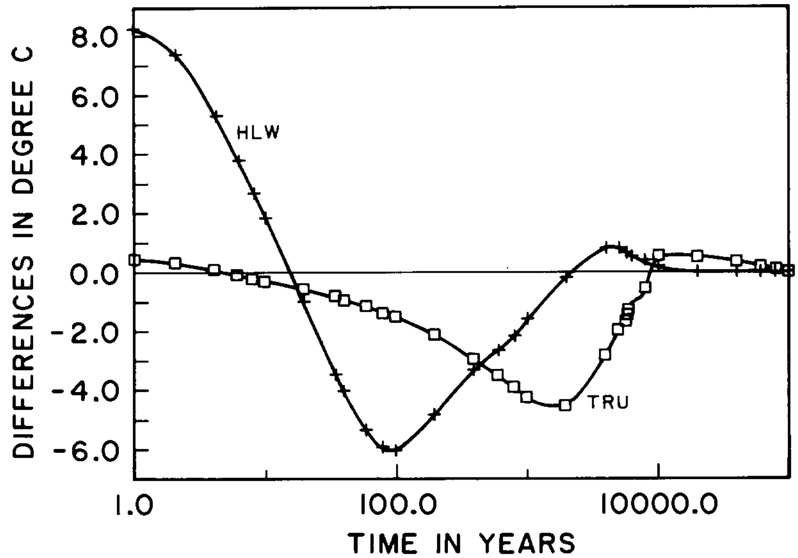

Fig. 4. Differences of the exact and numerical solutions of Figs. 2 and 3.

In Fig. 3 TRU waste is considered. It shows that the thermal response of the two cases are similar to the ones of Fig. 2. However, the differences in temperature are smaller, as can be seen in Fig. 4. For case B, ΔT(max) was 80.6°C. The maxima occur approximately at the same time (5700 years for case B and 6100 years for case A). The difference in temperatures represent an error of 1.6%. The differences between the solutions also were plotted in Fig. 4. In this case, in the time interval between 5 years and 9000 years, case A had a lower temperature than case B. This is similar to the curve obtained for HLW.

From the above results, it is reasonable to state that the theoretical approximation, case A, to the more realistic case, case B, is good within an error of 1.6% and 4% for TRU and HLW respectively. This is the worst case; for other depths, where the temperature differences are smaller, the errors are also smaller. Also, it is observed that in both nuclear wastes considered, for the region where the maximum temperature rise occurs, case A acts as a lower limit of the most realistic case, case B.

It is important to note that when equation (4) is used a large error is introduced to the model of the density-specific heat product. That is the case, because the actual behavior of that product is almost constant for the geological media considered in this paper (rock salt). The error is as large as 48%. However, the maximum error in the temperature rise of case A with respect to case B was as low as 4% for HLW and 1.6% for TRU, as just stated.

6. Conclusion

The analytical solution of problem A was shown to be a good approximation to the numerical solution of problem B. The importance of the analytical solution, equations (9) and (10), is that it gives an explicit relation between the main parameters that control the thermal behavior of a NWR. It can be used for design purposes.

7. Nomenclature

c_p Constant-pressure heat, J/kg °K
C Density-specific heat product, J/m^3 °K
$F(T)$ Function of T
k Thermal conductivity, w/m °K
l Thickness of geological layer, m
n Natural number
p Exponent in a polynomial expression
$\ddot{q}$ Heat flux, w/m^2
t Time, years
T Temperature, °K
T_a Reference temperature for k, °K
T_r Reference temperature for $V(T)$, °K
V Transformed temperature, w/m
x Spatial coordinate, m

Greek symbols

α Thermal diffusivity, m^2/yrs
λ Decay constant, yrs^{-1}
γ_n Expansion coefficient, m^{-1}
ϱ Density, kg/m^3

Subindex

n Index, $n = 1, 2, 3, \ldots$
0 Reference value at $t = 0$
l Reference value at $x = l$

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